2nd Annual

Math Invitational

4Girls

Saturday, April 23, 2016

Sponsored By

Connection
TUTORING CENTER

Individual Round Solutions

1. Solve for x: $0.5(12x + 400) = 600 + 4 \times 200$

   (A) -200 **(B)** -100 **(C)** 0 **(D)** 100 **(E)** 200

   Answer: (E)
   Solution(s)

$$\begin{aligned} 0.5(12x + 400) &= 600 + 4 \times 200 \\ 6x + 200 &= 600 + 800 \\ 6x &= 1200 \\ x &= \boxed{200 \quad (E)} \end{aligned}$$

2. A right triangle has a hypotenuse of length 15 and a leg of length 12. What is the measure of its shorter leg?

   (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

   Answer: (D)
   Solution(s)
   Let the shorter leg be x. By the Pythagorean theorem, we have $x = \sqrt{15^2 - 12^2} = \sqrt{81} = \boxed{9(D)}$

3. Jonathan is thinking of a number. He multiplies his number by four, then adds eight to the product and yields twenty-four. Bob is thinking of a different number that is two more than Jonathan's number. What is Bob's number?

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

   Answer: (C)
   Solution(s)
   Let Jonathan's number be J. Then we have

$$\begin{aligned} 4J + 8 &= 24 \\ 4J &= 16 \\ J &= 4 \end{aligned}$$

   We also know that Bob's number is two more than that $4 + 2 = \boxed{6(C)}$

4. A menu has 6 appetizers, 4 entrees, and 5 desserts. Maya is allergic to eggs, and one appetizer and one dessert contain eggs. If a three-course meal consists of one appetizer, one entree, and one dessert, given Maya can order the rest of the dishes, how many different three-course meals could she order?

   (A) 40 **(B)** 60 **(C)** 80 **(D)** 100 **(E)** 120

   Answer: (C)
   Solution(s)
   Maya is allergic to one appetizer and one desert. That leaves 5 appetizers, 4 entrees, and 4 desserts. Therefore she can pick any one of the 5 appetizers, and one of the 4 entrees, and any one of the 4 desserts. She therefore has $5 \times 4 \times 4 = \boxed{80(C)}$ choices.

5. What is the sum of the first ten odd integers?

   (A) 81 **(B)** 100 **(C)** 121 **(D)** 144 **(E)** 169

   Answer: (B)
   Solution(s)
   The sum of the first n odd integers is given by n^2. Therefore our answer is $10^2 = \boxed{100(B)}$

6. A drawer contains twelve green socks and eight blue socks. Swathi draws one sock at a time at random without replacement. What is the least number of socks Swathi must draw in order to ensure she draws a green sock?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** It is impossible to ensure she draws a green sock

Answer: (C)
Solution(s)
Let us say she first draws the 8 blue socks. This is the worse possible scenario. Then, there are only green socks left. Therefore she only must draw 1 more to get a green sock, giving a total of $\boxed{9(C)}$ draws.

7. A box containing a shipment of Rubiks cubes is 12 centimeters long, 9 centimeters wide, and 6 centimeters tall. The box contains 5 Rubiks cubes, each with sides of length 3 centimeters. What is the volume of the space in the box unoccupied by the Rubiks cubes?

(A) 513 **(B)** 540 **(C)** 567 **(D)** 594 **(E)** 621

Answer: (A)
Solution(s)
We find the volume of the box

$$12 \times 9 \times 6 = 12 \times 54 = 540 + 108 = 648$$

We then find the volume occupied by the Rubik's cubes

$$5 \times 3 \times 3 \times 3 = 5 \times 27 = 135$$

Subtracting we have

$$648 - 135 = \boxed{513(A)}$$

8. Anita drove to work at 40 miles per hour, and the distance from her house to her workplace is 40 miles. On the way back home, however, she was in a hurry and drove at 60 miles per hour. Anita got caught for speeding and spent 20 extra minutes waiting for her speeding ticket. What was Anita's average speed for her round-trip? (Note: After getting her speeding ticket, Anita continued at the same rate of 60 miles per hour.)

(A) 36 **(B)** 40 **(C)** 48 **(D)** 50 **(E)** 54

Answer: (B)
Solution(s)
We need to find her total time for the trip. On the way there she spend $40/40 = 1$ hour driving. On the way back she spends 20 min getting a speeding ticket, but drives at 60 mph the whole distance. $40/60 = 2/3$ hours, or 40 minutes driving. Adding these up, she spends 60 minutes or another hour on the way back. This means she spends 2 hours total, giving an average speed of $80/2 = \boxed{40(B)}$.

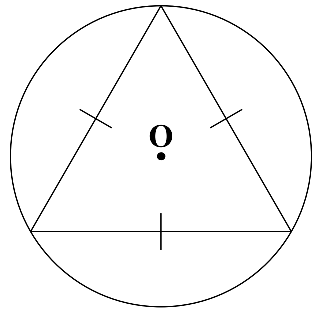
9. Lavender flips a fair-sided coin once. If it lands on heads, she will draw a marble from Bag A. If it lands on tails, she will draw a marble from Bag B. Bag A contains five scarlet marbles and three violet marbles. Bag B contains nine scarlet marbles and eleven violet marbles. What is the probability Lavender draws a violet marble?

(A) $\frac{37}{80}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{43}{80}$ **(D)** $\frac{23}{40}$ **(E)** $\frac{49}{80}$

Answer: (A)
Solution(s)
Lavender has $\frac{1}{2}$ chance of drawing from either bag. Let us say she draws from Bag A first. Then, she will have a $\frac{3}{3+5} = \frac{3}{8}$ chance of drawing a violet marble. This will be multiplied by $\frac{1}{2}$ because she had a $\frac{1}{2}$ chance of going to Bag A. Likewise, the probability of her drawing a voilet marble from Bag B would be $\frac{11}{20}$. This would also be multiplied by $\frac{1}{2}$. Therefore, the total probability is

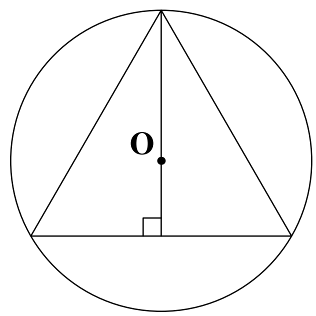
$$\frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{11}{20} = \frac{3}{16} + \frac{11}{40} = \boxed{\frac{37}{80}(A)}$$

10. What is the area of the region inside circle but outside the equilateral triangle with sides of length 6?



(A) $12\pi - 6\sqrt{3}$ **(B)** $12\pi - 9\sqrt{3}$ **(C)** $9\pi - 6\sqrt{3}$ **(D)** $9\pi - 9\sqrt{3}$ **(E)** 9π

Answer: (B)
Solution(s)
We drop an altitude down from the upper vertex to create a 30-60-90 triangle. This altitude is then also a median because the triangle is equilateral. Then, O is the centroid as it is also the circumcenter and the triangle is equilateral.

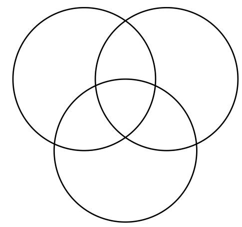


We have a right triangle with a hypotenuse of length 6, and a leg of length $6/2 = 3$ (remember that the altitude bisected the side). Therefore, the second leg has length $\sqrt{36-9} = \sqrt{27} = 3\sqrt{3}$. The radius is $2/3$ this, as the distance from O to the upper vertex is the radius, and this length is 2/3 the altitude. This definition results from the fact that O is the centroid, which divides the median (which is also an altitude) into 2 parts: a smaller one that is 1/3 the altitude/median, and one that is 2/3 it. The 2/3 piece is also the radius, as it is the distance from a point on the circle to the center. Thus, the radius has length $2\sqrt{3}$. The area of the circle is then $(2\sqrt{3})^2\pi = 12\pi$. We then subtract the area of the inner triangle, which is equilateral. The area of an equilateral triangle is given by $s^2\frac{\sqrt{3}}{4}$. Plugging in 6 we get $9\sqrt{3}$. Therefore, we subtract the 2 final quantities to get $\boxed{12\pi - 9\sqrt{3}(B)}$.

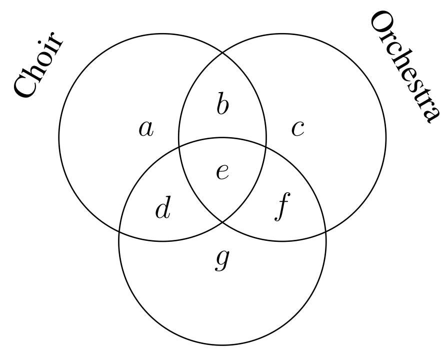
11. Music Masters Middle has a total of 105 students. 45 students are singers in Choir, 70 play instruments in Orchestra, and 53 are pianists in the Beethoven Club. If 25 are in both Choir and Orchestra, 33 are in both Orchestra and the Beethoven Club, and 20 are in both Choir and the Beethoven Club, how many students are members of all three musical groups?

(A) 0 **(B)** 5 **(C)** 10 **(D)** 15 **(E)** 20

Answer: (D)
Solution(s)
One of the best ways to solve this question is to use a venn diagram. Let us create the following venn diagram. Each of the larger circles indicates a musical group, while their intersections indicate people in more than one group.



We can label each region:



We know that there are 45 students in choir, so

$$a + b + e + d = 45 \tag{1}$$

We know that there are 70 students in orchestra, so

$$b + e + f + c = 70 \tag{2}$$

53 in Beethoven club:

$$d + e + f + g = 53 \tag{3}$$

25 in both choir and orchestra:

$$b + e = 25 \tag{4}$$

33 in orchestra and Beethoven club

$$e + f = 33 \tag{5}$$

20 in both choir and Beethoven club:

$$d + e = 20 \tag{6}$$

105 students in all:

$$a + b + c + d + e + f + g = 105 \tag{7}$$

Let us add equations (1), (2), and (3):

$$\begin{aligned} a + b + e + d &= 45 \\ b + e + f + c &= 70 \\ d + e + f + g &= 53 \\ a + b + e + d + b + e + f + c + d + e + f + g &= 168 \\ a + 2b + c + 2d + 3e + 2f + g &= 168 \end{aligned} \tag{8}$$

Let us add equations (4), (5), and (6):

$$\begin{aligned} b+e &= 25 \\ e+f &= 33 \\ d+e &= 20 \\ b+f+d+3e &= 78 \end{aligned} \tag{9}$$

Subtracting (9) from (8):

$$\begin{aligned} a+2b+c+2d+3e+2f+g &= 168 \\ b+f+d+3e &= 78 \\ a+b+c+d+f+g &= 90 \end{aligned} \tag{10}$$

Subtracting (10) from (7):

$$\begin{aligned} a+b+c+d+f+g &= 90 \\ a+b+c+d+e+f+g &= 105 \\ e &= 15 \end{aligned} \tag{11}$$

Notice that e is exactly the piece we need: the spot where students are in all 3 clubs. Therefore the answer is $\boxed{15(D)}$

12. Find the measure of the smaller angle formed by the hour and minute hand of a clock at 6:45. *Note: Question was discarded during competition; all answer choices were to be divided by 2*

(A) $50°$ **(B)** $60°$ **(C)** $67.5°$ **(D)** $75°$ **(E)** $87.5°$

Answer: (C)
Solution(s)
Using the formula for the angle formed by the hands of a clock, we get:

$$|\frac{1}{2}(60H + 11M)| \tag{12}$$

where H is the hour and M is the minutes past the hour.

$$\begin{aligned} \left|\frac{1}{2}(60(6) - 11(45))\right| &= \left|\frac{1}{2}(360 - 495)\right| \\ &= \left|\frac{1}{2}(-135)\right| \\ &= |-67.5| \\ &= \boxed{67.5(C)} \end{aligned} \tag{13}$$

13. Aimée thoroughly and evenly mixes a glass of 15 ounces of milk and 10 ounces of coffee. However, her friend Jaime drinks ten ounces. What is the absolute value difference between the number of ounces of milk and ounces of coffee Aimée needs to add to the mixture now in order to achieve a 30 ounce beverage that is 70% coffee and 30% milk? (E.g. The absolute value difference of 5 - 7 is 2.)

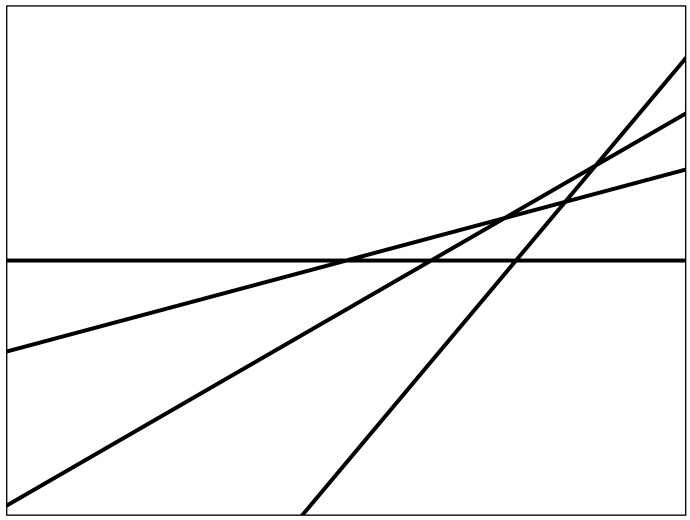
(A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

Answer: (C)
Solution(s)
If the mixture has 15 oz of milk and 10 oz of coffee, then $\frac{3}{5}$ is milk and $\frac{2}{5}$ is coffee. Therefore, when Jaime drinks 10 oz, he drinks $\frac{3}{5} \times 10 = 6$ oz of milk and $\frac{2}{5} \times 10 = 4$ oz of coffee. There are now $15 - 6 = 9$ oz of milk left and $10 - 4 = 6$ oz of coffee left.
In a 30 oz beverage that is 70% coffee and 30% milk, there are $30 \times 70\% = 21$ oz of coffee and $30 \times 30\% = 9$ oz of milk. Therefore, she must add $21 - 6 = 15$ oz of coffee and $9 - 9 = 0$ oz of milk. $|15 - 0| = \boxed{15(C)}$

14. What is the maximum number of regions that can be formed with 4 lines on a plane?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Answer: (B)
Solution(s)



As seen above, there is a maximum of $\boxed{11(B)}$

15. Whole numbers k and m satisfy $k \times m = 60$. Find the largest possible value of $k + m$.

(A) 17 **(B)** 19 **(C)** 23 **(D)** 32 **(E)** 61

Answer: (E)
Solution(s)
The maximum sum occurs when $k = 60$ and $m = 1$ or when $m = 60$ and $k = 1$. Either way, the sum is $\boxed{61(E)}$

16. The equation of a parabola is $y = 3x^2 + 8x + 5$. The line $y = 8x + 80$ intersects this parabola at exactly two points. What is the area of the triangle with vertices at these two points of intersection and the origin?

(A) 250 **(B)** 300 **(C)** 350 **(D)** 400 **(E)** 450

Answer: (D)
Solution(s)
We set the 2 equations equal to find the points of intersection:

$$\begin{aligned} 3x^2 + 8x + 5 &= 8x + 80 \\ 3x^2 - 75 &= 0 \\ x^2 - 25 &= 0 \\ (x-5)(x+5) &= 0 \\ x = 5 &\text{ or } x = -5 \end{aligned}$$

We substitute these values back in to find the coordinates:
-5: $8 \times -5 + 80 = 40 \implies (-5, 40)$
5: $8 \times 5 + 80 = 120 \implies (5, 120)$

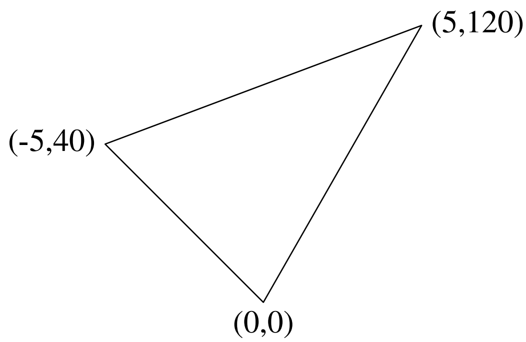
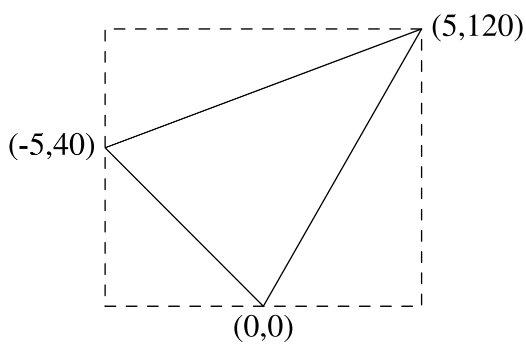


Figure is not to scale

We can draw a "box" around this



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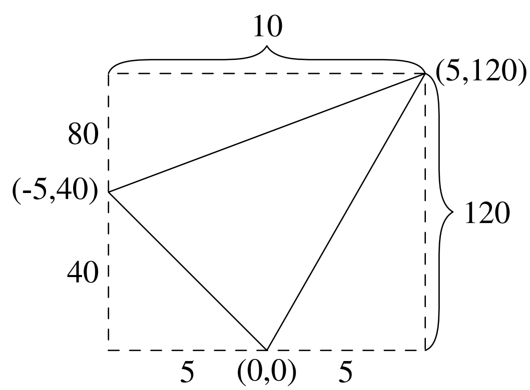
We have then created a bunch of right triangles, and can find the area of each of them. We can then subtract their areas from the area of the total box to find the area of our triangle.



The total area of all of the right triangles is then $\frac{1}{2}(40 \times 5 + 80 \times 100 = +120 \times 5) = \frac{1}{2}(200 + 800 + 600) = \frac{1}{2}(1600) = 800$. The total area of the box is $10 \times 120 = 1200$, making the area of the triangle equal to $1200 - 800 = \boxed{400(D)}$

17. Emi keeps rolling a dice until she rolls a two. What is the probability it takes her at least three rolls before she rolls a two? (E.g. Three rolls would constitute of two rolls of a number other than two and the last roll of a two.)

(A) $\frac{1}{6}$ **(B)** $\frac{5}{6}$ **(C)** $\frac{5}{36}$ **(D)** $\frac{25}{36}$ **(E)** 1

Answer: (D)
Solution(s)

Let us consider the probability of her NOT taking 3 or more rolls to get a 2, and subtract it from 1:
1 roll: The probability is $\frac{1}{6}$
2 rolls: The probability is $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$ Adding:

$$\frac{1}{6} + \frac{5}{36} = \frac{11}{36}$$

. Therefore, the total probability is

$$1 - \frac{11}{36} = \boxed{\frac{25}{36}(D)}$$

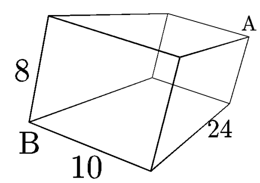
.

18. Find the sum of the coefficients of the polynomial $(x-1)(x-3)(x-5)(x-7)(x-9)$.

(A) -384 **(B)** -192 **(C)** 0 **(D)** 192 **(E)** 384

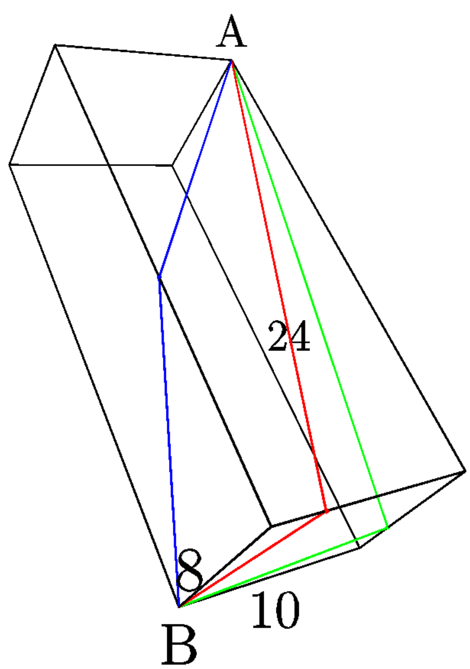
Answer: **(C)**
Solution(s)
Note that if we substitute 1 for x, all instances of x will be removed in the expansion, leaving only the coefficients. Therefore, the answer is $0 \times -2 \times -4 \times -6 \times -8 = \boxed{0(C)}$

19. A worm is trying to crawl from Corner A of a room to Corner B. The room is a rectangular prism with edges of lengths 8 ft, 10 ft, and 24 ft, as shown to the right. What is the shortest distance in feet the worm must crawl to reach its destination? Assume the worm can crawl across the edges and faces of the room.



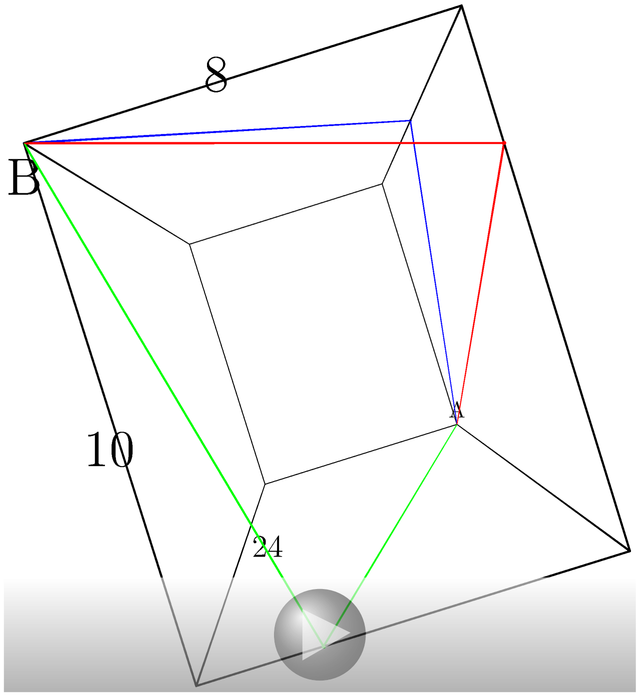
(A) 12 **(B)** 26 **(C)** $\sqrt{740}$ **(D)** 30 **(E)** 42

Answer: **(D)**
3 possible paths that could be the shortest:



For each of these note that if we "unravel" the cube, we create a right triangle.
Blue: One leg of the triangle has length 8+10, and the other has length 24. Therefore, the total length is $\sqrt{(8+10)^2 + 24^2} = \sqrt{18^2 + 24} = \sqrt{900} = 30$.
Red: $\sqrt{(8+24)^2 + 10^2} = \sqrt{32^2 + 100} = \sqrt{1124}$

Green: $\sqrt{(10+24)^2+8^2} = \sqrt{34^2+8^2} = \sqrt{1220}$ It is now plain to see that the smallest length is $\boxed{30(D)}$ An interactive view of the figure is shown here:



20. What is the sum of all positive integer factors of 2016?

(A) 6552 **(B)** 6656 **(C)** 7056 **(D)** 7371 **(E)** 8064

Answer: (A)
Solution(s)
To find the sum of all the positive integer factors of a number, you take each of the number's prime factors and add each of its powers starting from 0 and ending at its power in the number's prime factorization. Mathematically stated, for integer $A = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4} \cdots p_k^{a_k}$, the sum of its factors is

$$\prod_{i=1}^{k} \left(\sum_{j=1}^{a_i} p_i^j \right)$$

We factorize 2016 to be $2^7 \times 3^2 \times 7$. Therefore its sum of factors would be

$$(2^0+2^1+2^2+2^3+2^4+2^5+2^6+2^7)(3^0+3^1+3^2)(1+7) = \boxed{6552(A)}$$

21. How many zeros are at the end of the product
$125 \times 125 \times 25 \times 25 \times 5 \times 9 \times 9 \times 8 \times 8 \times 4 \times 4 \times 4 \times 2$

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Answer: (A)
Solution(s)
In order to find the number of zeros at the end of the number, we must find how many powers of 10 it has. Note that $10 = 2 \times 5$. Therefore we must count the number of powers of 2 and 5. Starting with 5, we note that $125 = 5^3$, $25 = 5^2$, and $5 = 5^1$. There are 2 125's, 2 25's, and 1 5. That gives a total of 6+4+1 = 11 factors of 5. Then, we note that $8 = 2^3$, $4 = 2^2$, and $2 = 2^1$. We have 2 8's, 3 4's, and 1 2. This gives a total of 6+6+1 = 13 factors of 2. Therefore we have $2^{13} \times 5^{11} = (2 \times 5)^1 1 \times 2^2 = 10^1 1 \times 2^2$, giving $\boxed{11(A)}$ powers of 10.

22. Find x if $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$

(A) $\sqrt{2}$ **(B)** 0.5 **(C)** 1 **(D)** 2 **(E)** none of these

Answer: (D)
Solution(s)

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$
$$x^2 = 2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

note that this second part is the same as the original x

$$x^2 = 2 + x$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

From here we have $x = 2$ or $x = -1$. We take the first solution because a square root cannot be negative giving the answer to be $\boxed{2(D)}$.

23. A number which when divided by 10 leaves a remainder of 8, when divided by 9 leaves a remainder of 7, when divided by 8 leaves a remainder 6, etc., down to where, when divided by 2, it leaves a remainder of 0, is:

(A) 418 **(B)** 1258 **(C)** 2518 **(D)** 3508 **(E)** $none of these$

Answer: (C)
Solution(s)
Let our number be x. Using the statements given in the question, we proceed with modular arithmetic:

$$\begin{aligned}
x &\equiv 8 \mod 10\\
x &\equiv 7 \mod 9\\
x &\equiv 6 \mod 8\\
x &\equiv 5 \mod 7\\
x &\equiv 4 \mod 6\\
x &\equiv 3 \mod 5\\
x &\equiv 2 \mod 4\\
x &\equiv 1 \mod 3\\
x &\equiv 0 \mod 2
\end{aligned}$$

Note that if we add 2 to each of the equations, all of the right hand sides become $0 \mod n$.

$$\begin{aligned}
x+2 &\equiv 0 \mod 10\\
x+2 &\equiv 0 \mod 9\\
x+2 &\equiv 0 \mod 8\\
x+2 &\equiv 0 \mod 7\\
x+2 &\equiv 0 \mod 6\\
x+2 &\equiv 0 \mod 5\\
x+2 &\equiv 0 \mod 4\\
x+2 &\equiv 0 \mod 3\\
x+2 &\equiv 0 \mod 2
\end{aligned}$$

Therefore, $x+2 = \gcd(2,3,4,5,6,7,8,9,10)$. This number is $5 \times 7 \times 8 \times 9 = 2520$. We subtract 2 from this to get $\boxed{2518(C)}$

24. Find the sum of all positive integers n such that $30 + n^2$ will be a perfect square.

(A) 14 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** none of these

Solution(s)
Let this perfect square be m^2, where m is an integer. Then we have:

$$30 + n^2 = m^2$$
$$m^2 - n^2 = 30$$
$$(m-n)(m+n) = 30$$

Knowing that n is positive, $m+n > m-n$, so we can proceed with the following generic system of equations (here $a \times b = 30$):

$$\begin{cases} m - n = a \\ m + n = b \end{cases}$$

Subtracting the second from the first gives

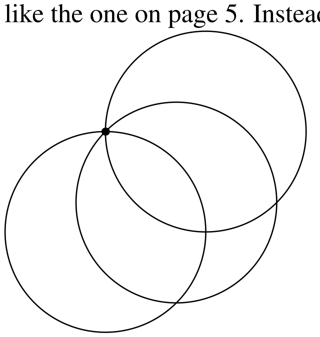
$$2n = b - a$$

yielding $n = \frac{b-a}{2}$, showing that the difference between factors must be even. However, note that 30 has only one factor of 2, meaning that $b - a$ can never be even.

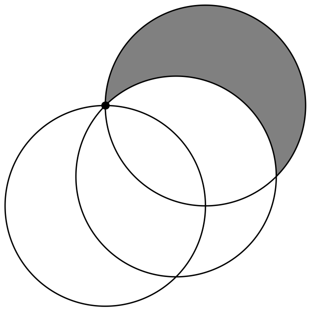
25. Mimi draws 150 congruent circles in the plane, all passing through a fixed point K. What is the largest number of regions into which these circles can split the plane? (Include the region outside the circles in your count.)

(A) $10,879$ **(B)** $11,027$ **(C)** $11,176$ **(D)** $11,326$ **(E)** $11,477$

Answer: (D)
Solution(s)
Let us start small, and not consider the outside region first. Let us say we have only 2 circles. The first circle would create 1 region. Then adding a second would create 2 more; the circle itself and the intersection between the 2 circles. Then let us move on to 3 circles. We already have 3 regions. The new circle can intersect each of the previous circles, and each time it does so it creates a new region. Additionally, it could intersect the overlapping region. However, if it were to intersect the overlapping region, it would not be able to intersect the first circle, as all 3 circles must pass through a common point. This is the reason we cannot have a configuration like the one on page 5. Instead, it would look more like this:



Note that each time the new circle intersects a previous one, it creates a new region. Therefore, the n^{th} circle added creates $n-1$ circles. However, it also creates a region itself:



Therefore, the nthcircle creates $n-1+1=n$ new regions. We then want to find the sum:

$$1+2+3+4+\cdots+149+150$$

Using the formula for the sum of the first k integers, which is $\frac{(k)(k+1)}{2}$, we get $\frac{(150)(151)}{2}=75\times 151=11325$. We then add 1 to this because of the region included outside of the circles to get $\boxed{11326(D)}$.
Here is an image of the figure itself: