2nd Annual

Math Invitational

4Girls

Saturday, April 23, 2016

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Connection
TUTORING CENTER

Team Round
Solutions

1. ___6___ Johnny has 5 apples. His mother gives him 8 oranges. Johnny eats 5 of these oranges, and then gives his sister, Corrine, 2 of his apples. How many pieces of fruit does he have left?

Solution(s)

He first has 5 fruit, and then gains 8, so he then has 13. Afterwards, he eats 5, and then gives away 2, overall losing 7 fruit. $13 - 7 = \boxed{6}$

2. ___8___ Haysonne has a number of quarters, dimes, nickels, and pennies, which total to $2.05. If his dimes were quarters and his quarters dimes, and his nickels were pennies and his pennies nickels, his total would be $2.23. He has two more nickels than dimes and one more penny than quarters. How many nickels does Haysonne have?

Solution(s)

Let q be the number of quarters, d the number of dimes, n the number of nickels, and p the number of pennies. Then, from the statement

> "...number of quarters, dimes, nickels, and pennies, which total to $2.05"

we know that

$$0.25 \times q + 0.10 \times d + 0.05 \times n + 0.01 \times p = 2.05$$

Multiplying this by 100 gives

$$25q + 10d + 5n + p = 205$$

Then, from the statement

> "If his dimes were quarters and his quarters dimes, and his nickels were pennies and his pennies nickels, his total would be $2.23."

we know that if we exchange the values of quarters and dimes, and exchange the values of nickles and pennies, we can have another equation:

$$0.25 \times d + 0.10 \times q + 0.05 \times p + 0.01 \times n = 2.05$$

Simplifying yields

$$25d + 10q + 5p + n = 223$$

From the statements

> "He has two more nickels than dimes"

and

> "He has...one more penny than quarters"

we can write the equations

$$n = d + 2$$

and

$$p = q + 1$$

Now we have 4 equations for 4 variables, and therefore a system of equations:

$$\begin{cases} 25q + 10d + 5n + p = 205 \\ 25d + 10q + 5p + n = 223 \\ n = d + 2 \\ p = q + 1 \end{cases}$$

From here we can then proceed in several ways:

Method 1: Traditional Method

Here we use a combination of substitution and elimination to solve the system. Remember that we wish to solve for the number of nickels, so it would be good to solve for them directly. We proceed by rewriting one equation such that we can directly solve for nickels:

$$\begin{cases} 25q + 10d + 5n + p = 205 & (1) \\ 25d + 10q + 5p + n = 223 & (2) \\ n - 2 = d & (3) \\ p = q + 1 & (4) \end{cases}$$

We substitute (3) and (4) each into (1) and (2):

$$\begin{cases} 25q + 10(n-2) + 5n + (q+1) = 205 & (5) \\ 25(n-2) + 10q + 5(q+1) + n = 223 & (6) \end{cases}$$

We simplify each:

$$\begin{aligned} 25q + 10(n-2) + 5n + (q+1) &= 205 \\ 25q + 10n - 20 + 5n + q + 1 &= 205 \\ 26q + 15n - 19 &= 205 \\ 26q + 15n &= 224 \end{aligned} \tag{7}$$

$$\begin{aligned} 25(n-2) + 10q + 5(q+1) + n &= 223 \\ 25n - 50 + 10q + 5q + 5 + n &= 223 \\ 26n + 15q - 45 &= 223 \\ 26n + 15q &= 268 \end{aligned} \tag{8}$$

Now we have reduced it to a 2 variable system:

$$\begin{cases} 26q + 15n = 224 & (9) \\ 15q + 26n = 268 & (10) \end{cases}$$

Now we multiply (7) by 15 and (8) by 26 to get $15(26q + 15n) = 15 \times 224$ and $26(15q + 26n) = 26 \times 268$, respectively. We then simplify (remember you can use a calculator):

$$\begin{cases} 390q + 225n = 3360 & (11) \\ 390q + 676n = 6968 & (12) \end{cases}$$

We subtract (11) from (12) to get 1 equation, which we can solve for n:

$$\begin{aligned} 390q + 676n - 390q - 225n &= 6968 - 3360 \\ 451n &= 3608 \\ n &= \boxed{8} \end{aligned} \tag{13}$$

Method 2: Matrix Inversion
We rewrite the equations to be put into a matrix

$$\begin{cases} 25q + 10d + 5n + p = 205 \\ 10q + 25d + n + 5p = 223 \\ -d + n = 2 \\ -q + p = 1 \end{cases}$$

Place them into a matrix and then solve:

$$\begin{bmatrix} 25 & 10 & 5 & 1 \\ 10 & 25 & 1 & 5 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ d \\ n \\ p \end{bmatrix} = \begin{bmatrix} 205 \\ 223 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 10 & 5 & 1 \\ 10 & 25 & 1 & 5 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 25 & 10 & 5 & 1 \\ 10 & 25 & 1 & 5 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ d \\ n \\ p \end{bmatrix} = \begin{bmatrix} 25 & 10 & 5 & 1 \\ 10 & 25 & 1 & 5 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 205 \\ 223 \\ 2 \\ 1 \end{bmatrix}$$

You can use a calculator to find the inverse

$$\begin{bmatrix} q \\ d \\ n \\ p \end{bmatrix} = \begin{bmatrix} \frac{26}{451} & -\frac{15}{451} & -\frac{115}{451} & \frac{49}{451} \\ -\frac{15}{451} & \frac{26}{451} & \frac{49}{451} & -\frac{115}{451} \\ -\frac{15}{451} & \frac{26}{451} & \frac{500}{451} & -\frac{115}{451} \\ \frac{26}{451} & -\frac{15}{451} & -\frac{115}{451} & \frac{500}{451} \end{bmatrix} \begin{bmatrix} 205 \\ 223 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} q \\ d \\ n \\ p \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \\ 5 \end{bmatrix}$$

$$n = \boxed{8}$$

3. ___169___ Angela is thinking of a perfect square. Her number contains three distinct digits, and when the tens and ones digits are flipped, the resulting number is a greater perfect square. What perfect square is Angela thinking of?

Solution(s)

Begin listing out the perfect squares greater than 100:

11^2	121
12^2	144
13^2	169
14^2	196

Immediately we can notice that 169 and 196 fit the criteria. We need the smaller perfect square, sor our answer is $\boxed{169}$.

4. ___$9\frac{3}{8}$___ Luis is a glue manufacturer. He can make one whole block of glue in five hours by himself. Marco can manufacture the same amount in three hours. How long would it take them to manufacture five blocks of glue working together at their same usual rates? Express your answer as a mixed fraction.

Solution(s)

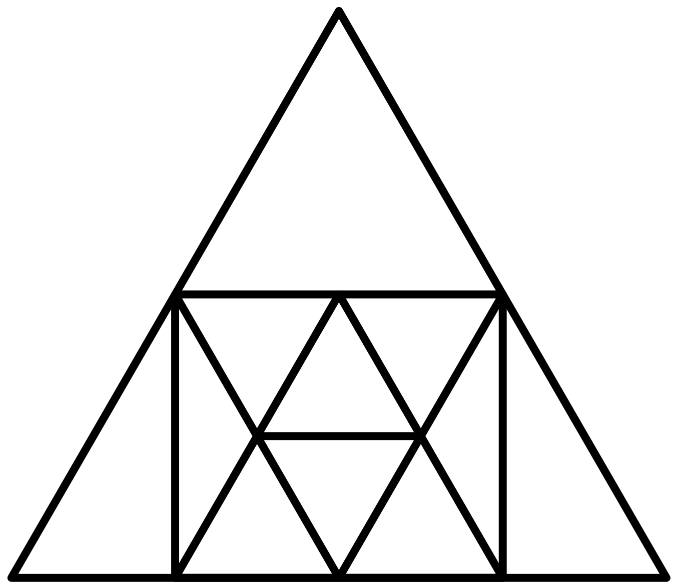
Luis's speed in blocks per hour: $\frac{1\ block}{5\ hours}$

Marco's speed in blocks per hour: $\frac{1\ block}{3\ hours}$

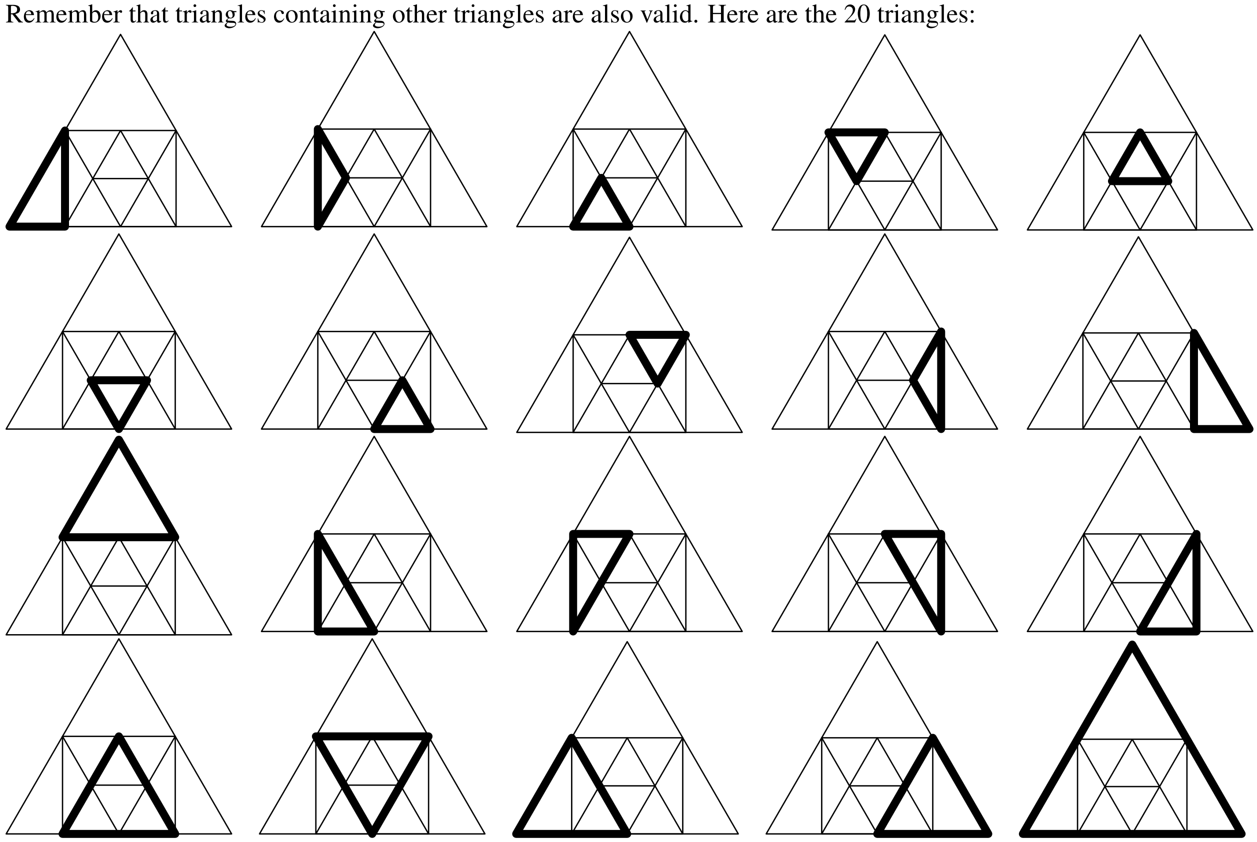
We add their speeds together to get $\frac{1}{5} + \frac{1}{3} = \frac{8\ blocks}{15\ hours}$. We the number of blocks they have to cut by this speed:

$$\frac{5}{\frac{8}{15}} = 5 \times \frac{15}{8} = \frac{75}{8} = \boxed{9\frac{3}{8}}$$

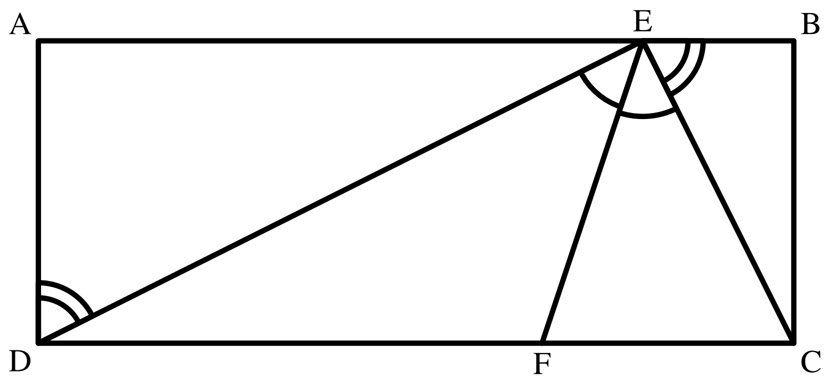
5. ___20___ How many triangles are in the following diagram?



Solution(s)
Remember that triangles containing other triangles are also valid. Here are the 20 triangles:

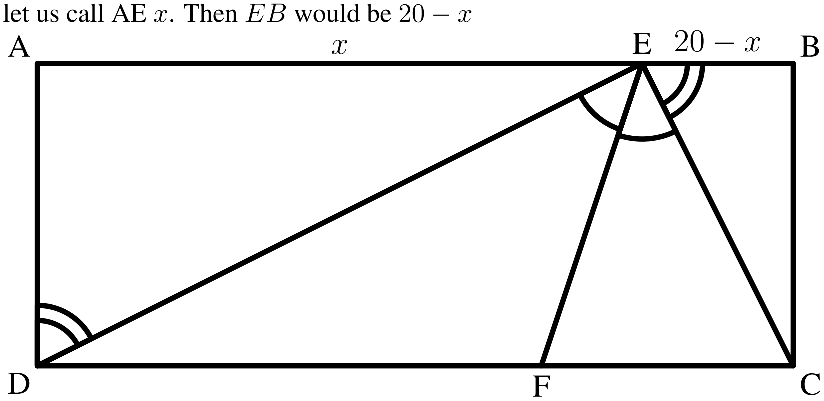


6. ___$\frac{800}{9}$___ In rectangle ABCD, the length of AD is 8 and the length of AB is 20. Angles ADE and BEC are congruent, and angles DEF and FEC are congruent. What is the length of DF multiplied by the length of FC? Express your answer as a common fraction.



Solution(s)

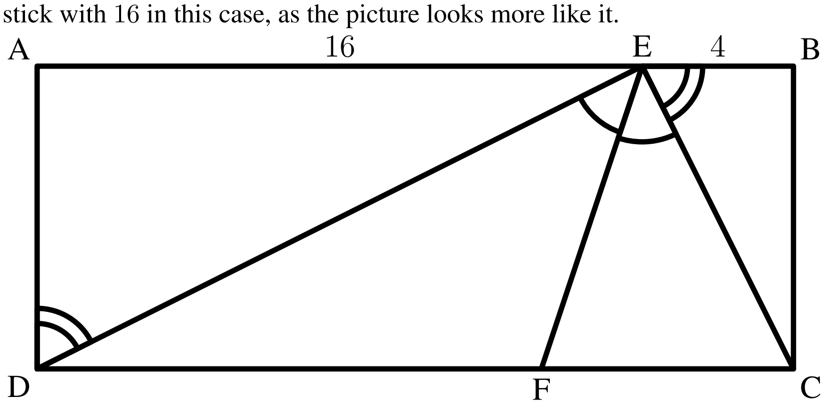
We know that $\triangle ADE$ and $\triangle BEC$ are similar because they are both right triangles and have 1 congruent angle. Then, let us call AE x. Then EB would be $20 - x$



Because the 2 triangles are similar, we can set up a proportion:

$$\frac{AD}{AE} = \frac{BE}{BC}$$
$$\frac{8}{x} = \frac{20 - x}{8}$$
$$64 = 20x - x^2$$
$$x^2 - 20x + 64 = 0$$
$$(x - 16)(x - 4) = 0$$

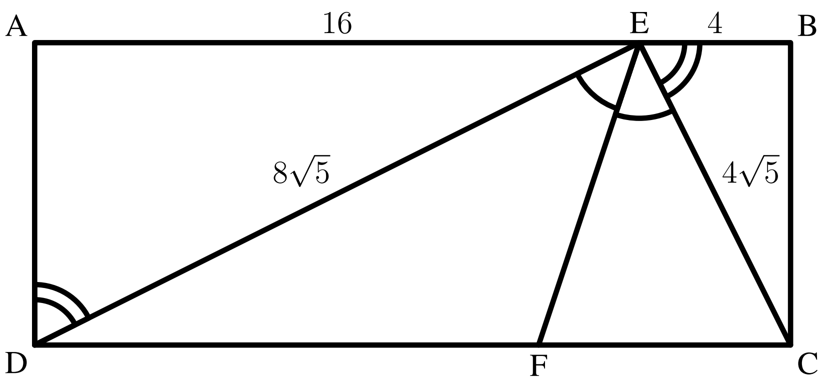
We now have the $x = 16$ or $x = 4$. However, we can take either one because (a) using 4 would just switch the lengths of AE and EB, and (b) the end asks for the product, which is commutative, meaning the order doesn't matter. We will stick with 16 in this case, as the picture looks more like it.



We now can find the lengths of DE and EC, and then proceed with the angle bisector theorem;

$$\begin{aligned} DE &= \sqrt{16^2 + 8^2} \\ &= \sqrt{256 + 64} \\ &= \sqrt{320} \\ &= 8\sqrt{5} \end{aligned}$$

$$\begin{aligned} EC &= \sqrt{4^2 + 8^2} \\ &= \sqrt{16 + 64} \\ &= \sqrt{80} \\ &= 4\sqrt{5} \end{aligned}$$



Let DF be y. It then follows that FC = $20 - y$. Using the angle bisector theorem, we have

$$\begin{aligned} \frac{DE}{EC} &= \frac{DF}{FC} \\ \frac{8\sqrt{5}}{4\sqrt{5}} &= \frac{y}{20 - y} \\ 2 &= \frac{y}{20 - y} \end{aligned}$$

Notice that this is also the ratio of AD to EB or AE to BC; this is true because corresponding sides of similar triangles are proportional; therefore, $\frac{AD}{EB} = \frac{AE}{BC} = \frac{DE}{EC}$

$$\begin{aligned} 40 - 2y &= y \\ 3y &= 40 \\ y &= \frac{40}{3} \\ 20 - y &= \frac{60}{3} - \frac{40}{3} \\ 20 - y &= \frac{20}{3} \end{aligned}$$

We now have that DF = $\frac{40}{3}$ and FC = $\frac{20}{3}$. Their product is then

$$\frac{40}{3} \times \frac{20}{3} = \boxed{\frac{800}{3}}$$

7. $\frac{35}{128}$ On Monday, John receives 100 dollars from his father. Over the next week, everyday there is a 50% chance he will receive a gift of 10 dollars from a magical Shamu. What is the probability that at the end of the week, John will have exactly 130 dollars?

Solution(s)

John must gain a total of 30 dollars over the next 7 days. Therefore, any of the scenarios that give this must have 3 days in which he gets money from the Shamu. Let G be a day that he gains money and N be a day nothing happens. Then, all of the combinations that have John gain 30 dollars must be a permutation of the following sequence: GGGNNNN. Using the formula for permutations with duplicates, we have $\frac{7!}{3! \times 4!}$, which gives 35.

The probability of any scenario happening is $\left(\frac{1}{2}\right)^8 = \frac{1}{128}$, so the result must be $35 \times \frac{1}{128} = \boxed{\frac{35}{128}}$

8. 5 John creates a grid that has 2 columns and 3 rows. He wants to place the numbers 1 through 6 in this grid, with the numbers strictly increasing downwards and to the left. How many distinct grids could he create?

Solution(s)

Here are the 5 grids:

1	2
3	4
5	6

1	3
2	5
4	6

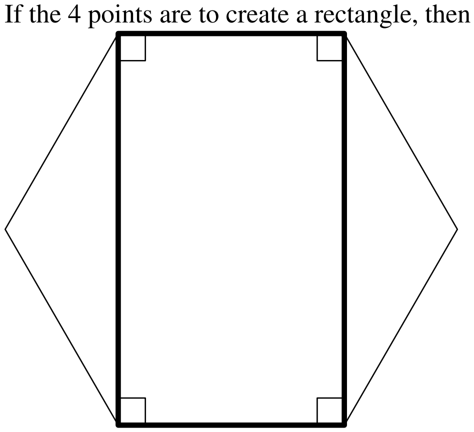
1	4
2	5
3	6

1	2
3	5
4	6

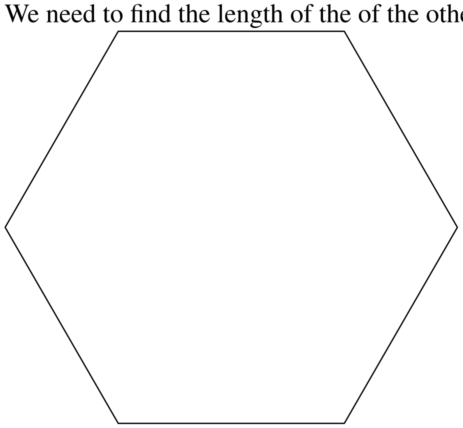
1	3
2	4
5	6

9. $9\sqrt{3}$ The four vertices of a rectangle are also the vertices of a regular hexagon of side length 3. What is the area of the rectangle? Express your answer in simplest radical form.

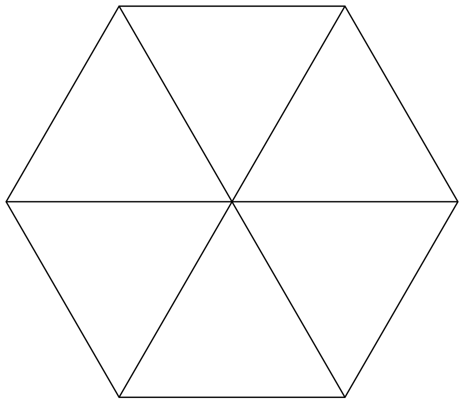
If the 4 points are to create a rectangle, then they must be spaced in the following fashion:



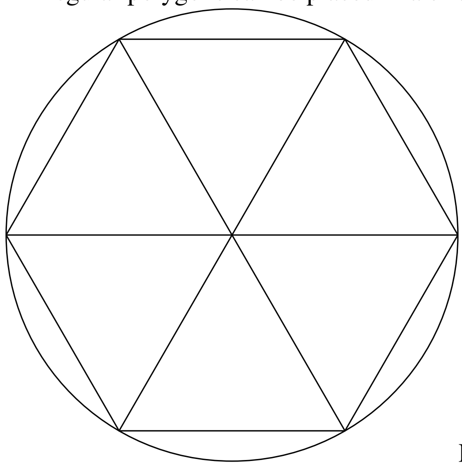
We need to find the length of the of the other side of the rectangle. To do this, let us first consider an empty hexagon.



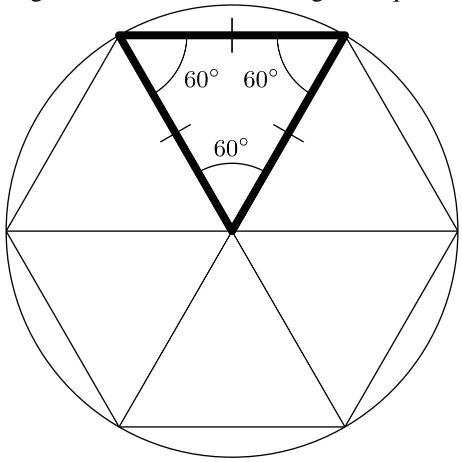
If we connect each vertex to the center, something special happens.



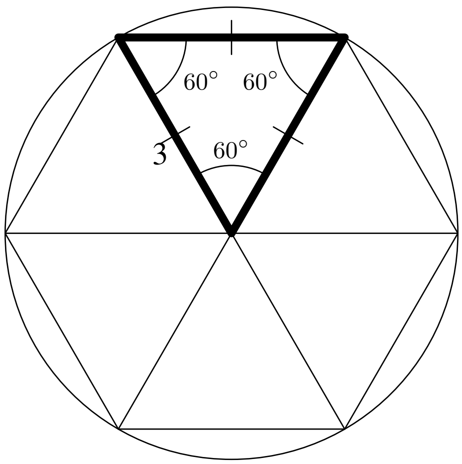
All regular polygons can be placed in a circle:



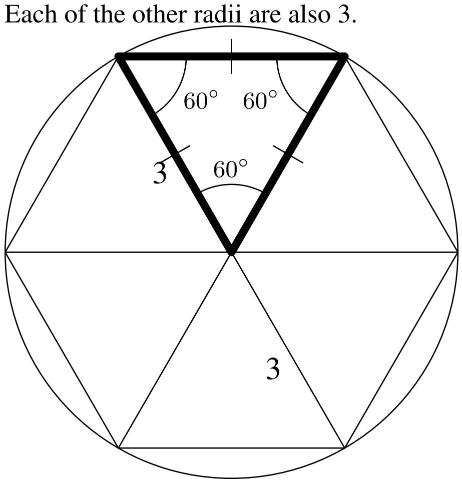
Each of the segments coming from the center is now a radius, and therefore they are all congruent. The central angle was once 360 degrees, but now we have cut it into 6 equal parts, so each now has measure 60 degrees. With 2 congruent sides, the triangle must be isosceles. The 60 degree angle is now the vertex angle, and therefore the triangle is equilateral.



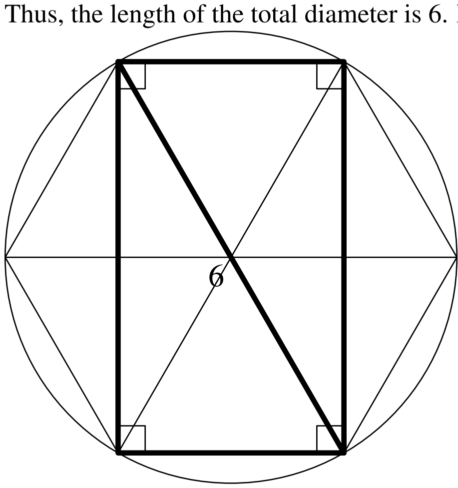
We already know the sidelength of the hexagon is 3, so therefore, the other sides of the equilateral triangle are also 3.



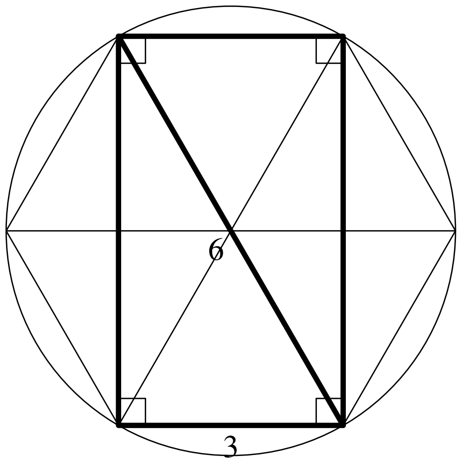
Each of the other radii are also 3.



Thus, the length of the total diameter is 6. Putting the rectangle back:



We also know the sidelength of the hexagon is 3.



We now have a right triangle with hypotenuse 6 and a leg of length 3. Therefore, the last leg has measure $\sqrt{36-9} = \sqrt{27} = 3\sqrt{3}$. We multiply this with 3 to get the area of the rectangle, giving $9\sqrt{3}$

10. 56 In Amy's area, phone numbers can have 6 digits with each digit ranging from 0 to 9. Bill can only remember 5 of the 6 digits of Amys phone number. Bill doesn't remember which digit he forgot nor its position in her phone number, but he remembers the order of the digits he does recall. How many different phone numbers would Bill have to dial in order to ensure that he dials Amy's number?

Let Amy's number be ABCDEF, where A-F are integers from 0-9. Bob forgets one of these numbers, and therefore we have 5 left. Let us represent thes 5 digits with _ _ _ _ _. Each line can have one number placed above it. Each of these numbers is already predetermined, as it was part of Amy's phone number. The number Bob forgot could go in 6 spots, shown here by $*$:

$$*\_*\_*\_*\_*\_*$$

Each of these $*$ can be any digit from 0-9, or 10 numbers. There are 6 spots, so a preliminary estimate gives $6\times 10 = 60$ phone numbers to dial. However, there is a special arrangement to consider. When we calculated 60, we assumed that when we put any of the digits 0-9 in each of the $*$ spots, we would create a new number. However, note the following case: Let the 5 digits Bob knows be $ABCDE$. The, we have $*A*B*C*D*E*$. Each of these letters is a digit 0-9, so let B be 5. Then, if we insert a 5 here: $*A5B*C*D*E*$, it is the same as inserting a 5 here: $*A*B5*C*D*E*$, as 5B and B5 are the same number. Therefore, for each of the numbers he knows, we can have this scenario, where the number we insert is equal to one of the 5 numbers we know. Therefore, we subtract 5 (the number of digits we know) from 60, to get an answer of $\boxed{55}$.

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